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A Proof of the Barát-Thomassen Conjecture*

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Abstract

The Barát-Thomassen conjecture asserts that for every tree T on m edges, there exists a constant k_T such that every k_T -edge-connected graph with size divisible by m can be edge-decomposed into copies of T . So far this conjecture has only been verified when T is a path or when T has diameter at most 4. Here we prove the full statement of the conjecture.

Keywords: tree-decomposition, large edge-connectivity, Barát-Thomassen Conjecture.

1 Introduction

Unless stated otherwise, all graphs considered in this paper are finite and simple. Given a graph G , we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively. For any subset S of vertices or edges of G , we denote by $G[S]$ the subgraph of G induced by S .

Given a connected graph H , we say that G has an H -decomposition if there exists a partition of $E(G)$ into parts isomorphic to H . When does a graph G admit an H -decomposition? In general, this question is NP-complete whenever H contains at least 3

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edges, see [7]. We focus on the case where we want to decompose G into copies of a given tree T . A necessary condition for the existence of a T -decomposition is of course that $|E(T)|$ divides $|E(G)|$. There are many theorems and conjectures in graph theory stating that this condition is also sufficient in certain cases. By a result of Wilson [17] this holds when G is a sufficiently large complete graph, and there exist more general results showing that this is also true for graphs of large minimum degree. More precisely, for every tree T there exists a constant $\varepsilon_T > 0$ such that every graph G of minimum degree $(1 - \varepsilon_T)|V(G)|$ admits a T -decomposition, provided its size is divisible by the size of T (see for example [2]).

A different line of research was started by Barát and Thomassen [3], when they observed in 2006 that T -decompositions are intimately related to nowhere-zero flows. Tutte conjectured that every 4-edge-connected graph admits a nowhere-zero 3-flow, but until recently it was not even known that any constant edge-connectivity suffices for this. Barát and Thomassen showed that if every 8-edge-connected graph of size divisible by 3 admits a $K_{1,3}$ -decomposition, then every 8-edge-connected graph admits a nowhere-zero 3-flow. Vice versa, they also showed that Tutte's 3-flow conjecture would imply that every 10-edge-connected graph with size divisible by 3 admits a $K_{1,3}$ -decomposition. Motivated by this intrinsic connection, they conjectured the following.

Conjecture 1 (Barát-Thomassen Conjecture, [3]). *For any tree T on m edges, there exists an integer k_T such that every k_T -edge-connected graph with size divisible by m has a T -decomposition.*

When the conjecture was made, it was only known to hold in the trivial cases where T has less than 3 edges. Since then, Conjecture 1 has attracted growing attention, and it has now been verified for different families of trees such as stars [14], some bistars [1, 16], and paths of a certain length [6, 12, 13, 15]. Very recently, breakthrough results were obtained by Merker [10], who proved the conjecture for all trees of diameter at most 4, hence covering some of the results above, and by Botler, Mota, Oshiro, and Wakabayashi [5], who proved the conjecture for all paths. The latter result was improved by Bensmail, Harutyunyan, Le, and Thomassé [4], who showed that, for path-decompositions, large minimum degree is a sufficient condition provided the graph is 24-edge-connected.

The purpose of this paper is to verify Conjecture 1 for every tree T , hence settling the conjecture in the affirmative.

Theorem 2. *The Barát-Thomassen conjecture is true.*

This paper builds upon previous work on the Barát-Thomassen conjecture. It was shown by Thomassen in [16], and independently by Barát and Gerbner in [1], that it is sufficient to verify Conjecture 1 for bipartite graphs G .

Theorem 3. [1, 16] *Let T be a tree on m edges. The following two statements are equivalent:*

- (1) *There exists a natural number k_T such that every k_T -edge-connected graph with size divisible by m has a T -decomposition.*
- (2) *There exists a natural number k'_T such that every k'_T -edge-connected bipartite graph with size divisible by m has a T -decomposition.*

An important tool in the study of Conjecture 1 is a recent result on modulo- k orientations, which was proved by Thomassen in [14]. He showed that the edges of a highly edge-connected graph can be oriented so that any prescribed out-degrees modulo k are realized. This was formerly known as Jaeger's weak k -flow conjecture and it immediately implies that Conjecture 1 holds for all stars. Moreover, his result implies that every 8-edge-connected graph admits a nowhere-zero 3-flow. In [8], the method was refined to show that also 6-edge-connected graphs admit a nowhere-zero 3-flow. The question whether edge-connectivity 4 suffices remains open. Another application of these modulo- k orientations is the following decomposition result, which was shown by Thomassen in [16] and also applied in [6] and [10].

Theorem 4. [16] *Let G be a bipartite graph with partition classes A_1 and A_2 , and size divisible by m . If G is $(4\lambda + 6m)$ -edge-connected, then G can be decomposed into two λ -edge-connected graphs G_1 and G_2 such that $d(v, G_i)$ is divisible by m for every v in A_i .*

By Theorems 3 and 4, it is sufficient to prove Conjecture 1 for bipartite graphs G on vertex classes A and B , where all vertices in A have degree divisible by m , the size of T .

Let T_A and T_B denote the vertex classes of a bipartition of T . We may assume that T_B contains a leaf. The T -decompositions we are going to construct will respect the bipartitions of G and T in the sense that the vertices corresponding to T_A will lie in A for each copy of T . We say that vertices $v \in V(G)$ and $t \in V(T)$ are *compatible* if $v \in A$ and $t \in T_A$, or $v \in B$ and $t \in T_B$.

We study a specific kind of edge-colouring of G , that was introduced in [10]. Assuming G is (improperly) edge-coloured, we denote by $d_i(v)$ the degree of vertex v in colour i . For $t \in V(T)$, let $S(t)$ denote the set of edges incident with t .

Definition 5. *An edge-colouring $\phi : E(G) \rightarrow E(T)$ is called T -equitable, if for any compatible vertices $v \in V(G)$, $t \in V(T)$ and $j, k \in S(t)$, we have $d_j(v) = d_k(v)$.*

It was shown by Merker [10], using modulo- k orientations, that highly edge-connected graphs admit T -equitable edge-colourings.

Theorem 6 (Theorem 3.4 in [10]). *For all natural numbers m and L there exists a natural number $f(m, L)$ such that the following holds: If G is an $f(m, L)$ -edge-connected bipartite graph with bipartite classes A and B where all vertices in A have degree divisible by m , and T is a tree on m edges with bipartite classes T_A and T_B where T_B contains a leaf, then G admits a T -equitable colouring where the minimum degree in each colour is at least L .*

Notice that since we put no constraints on the degrees in B , it is necessary that the greatest common divisor of the degrees in T_B is 1 if we want to construct a T -equitable colouring. For this reason we chose the bipartition of T so that T_B contains a leaf.

If there exists a T -decomposition of a bipartite graph G where all copies of T are oriented the same way (with respect to the bipartite classes), then this gives rise to a T -equitable colouring of G . Vice versa, a T -equitable colouring can also be used to construct a T -decomposition. This was done in [10] for the case that the girth of G is at least the diameter of T , and also in general for trees of diameter at most 4.

In this paper we use probabilistic methods, inspired from those used in [4], to show that a T -equitable colouring can be turned into a T -decomposition whenever the minimum degree in each colour is large enough.

Theorem 7. *Let T be a tree on m edges and let G be a bipartite graph admitting a T -equitable colouring. If the minimum degree in each colour is at least 10^{50m} , then G has a T -decomposition.*

Combined with the previous theorems, Theorem 7 completes the proof of the Barát-Thomassen conjecture.

Proof of Theorem 2: By Theorem 3, we may assume that G is bipartite. We show that every $(4f(m, 10^{50m}) + 6m)$ -edge-connected bipartite graph has a T -decomposition, where f is the function given by Theorem 6. By Theorem 4 we can decompose G into two spanning $f(m, 10^{50m})$ -edge-connected graphs G_1 and G_2 , such that in one side of the bipartition of each G_i all vertices have degree divisible by m . By Theorem 6, we can find a T -equitable colouring of G_i in which the minimum degree in each colour is at least 10^{50m} . This colouring can be turned into a T -decomposition by Theorem 7. \square

2 Definitions and sketch of proof

In our proof of Theorem 7, a T -decomposition of a graph G is obtained in two steps, which we describe more formally below. In the first step we construct a decomposition of G into so-called pseudo-copies of T , which are subgraphs that are in some sense homomorphic to T . Such a decomposition, which we call a T -pseudo-decomposition, can easily be obtained from a T -equitable colouring, see also [10]. Instead of choosing any such decomposition, we use probabilistic methods to find one in which the vast majority of pseudo-copies at every vertex are isomorphic to T . The details of this step can be found in Section 3. In the second step, we use these isomorphic copies to repair the non-isomorphic copies of T by making subgraph switches. While the switching itself is a deterministic operation, we again use probabilistic methods to find a suitable set of isomorphic copies. This part of the proof is detailed in Section 4.

Step 1: Finding a good T -pseudo-decomposition

Let G be a graph with a T -equitable colouring. Recall that $S(t)$ denotes the set of edges incident with a vertex $t \in V(T)$. Let us denote by $N_i(v)$ the set of edges coloured i incident with $v \in V(G)$. Furthermore, we set $N_{S(t)}(v) := \bigcup_{i \in S(t)} N_i(v)$ for every $t \in V(T)$ compatible with v . Since the edge-colouring of G is T -equitable, we have that $|N_{S(t)}(v)|$ is divisible by $|S(t)|$ for every compatible $v \in V(G)$, $t \in V(T)$. Thus, we partition $N_{S(t)}(v)$ into stars of size $|S(t)|$ that contain each of the colours in $S(t)$ exactly once. Let \mathcal{S} be the collection of stars we get after having done this for every $v \in V(G)$ and compatible $t \in V(T)$. Consider an auxiliary graph $G_{\mathcal{S}}$ whose vertices are the stars in \mathcal{S} , and where two vertices are joined by an edge whenever the corresponding stars have an edge in common. By construction, each connected component of $G_{\mathcal{S}}$ is a tree isomorphic to T . For every connected component in $G_{\mathcal{S}}$, we take the union of all the stars corresponding to it in G . It is easy to see that this decomposes G into parts of the same size as T . In fact even more is true: Each part is isomorphic to a graph obtained from T by identifying vertices. This motivates the following definition.

Definition 8. *A graph H is a **pseudo-copy** of T , if there exists a surjective graph homomorphism $h : V(T) \rightarrow V(H)$ that induces a bijection between $E(T)$ and $E(H)$.*

In other words, a graph H is a pseudo-copy of T , if it is isomorphic to a multigraph obtained from T by identifying vertices and keeping all edges. We also refer to pseudo-copies of T as *pseudo-trees*. The term *T -pseudo-decomposition* denotes a decomposition where each part is a pseudo-copy of T . Given a T -equitable colouring of G , the construction above results in a T -pseudo-decomposition of G .

Notice that it might be the case that a graph H can be considered as a pseudo-copy of T in different ways if there exists more than one homomorphism from T to H with the required properties. However, we will only consider homomorphisms that induce the same edge-colouring of H as the given T -equitable colouring. Furthermore, we only consider pseudo-copies of T in G that respect the bipartition in the sense that vertices corresponding to T_A always lie in A .

Let \mathcal{P} be a T -pseudo-decomposition of G . For every compatible $v \in V(G)$ and $t \in V(T)$, we denote by $N_{\mathcal{P}}(v|t)$ the set of pseudo-trees in \mathcal{P} in which v is the image of t . Let $d_{\mathcal{P}}(v|t) = |N_{\mathcal{P}}(v|t)|$. Clearly, for any two different vertices u and v of G , we have $N_{\mathcal{P}}(u|t) \cap N_{\mathcal{P}}(v|t) = \emptyset$. Notice also that

$$\bigcup_{v \in G} N_{\mathcal{P}}(v|t) = \mathcal{P}$$

for every $t \in V(T)$.

So far we have explained how T -pseudo-decompositions can be obtained from a T -equitable colouring. We denote such a resulting T -pseudo-decomposition \mathcal{P} of G by $\mathcal{H} \cup \mathcal{I}$, where \mathcal{I} denotes the collection of pseudo-copies that are isomorphic to T and \mathcal{H} denotes the collection of the remaining pseudo-copies.

If the minimum degree in each colour is large in the T -equitable colouring, then there are many possibilities at every vertex to decompose its incident edges into stars. Using probabilistic methods, we find a T -pseudo-decomposition where $d_{\mathcal{H}}(v|t) \leq \varepsilon d_{\mathcal{I}}(v|t)$ for some given $\varepsilon > 0$ and every compatible $v \in V(G)$, $t \in V(T)$. Now for every non-isomorphic copy $H \in N_{\mathcal{H}}(v|t)$, there are many copies isomorphic to T in $N_{\mathcal{I}}(v|t)$. We will use one of these isomorphic copies to improve the T -pseudo-decomposition by repairing H . This is done by a subgraph switch operation which is explained in more detail in Step 2. However, if the trees in $N_{\mathcal{I}}(v|t)$ overlap too much, then we might not be able to make any switch that improves the T -pseudo-decomposition. To avoid this, we need to find a large set of isomorphic copies in $N_{\mathcal{I}}(v|t)$ that pairwise intersect only in v . To measure how much the pseudo-trees in a T -pseudo-decomposition overlap, we use the following concept that was introduced in [4].

Definition 9. Let \mathcal{P} be a collection of pseudo-copies of T in G , and $v \in V(G)$ and $t \in V(T)$ be compatible vertices. The **conflict ratio** of v with respect to t , denoted by $\text{conf}_{\mathcal{P}}(v|t)$, is defined by

$$\text{conf}_{\mathcal{P}}(v|t) := \frac{\max_{u \neq v} |\{T \in N_{\mathcal{P}}(v|t) : u \in V(T)\}|}{d_{\mathcal{P}}(v|t)}.$$

Intuitively, $\text{conf}_{\mathcal{P}}(v|t)$ measures the maximum proportion of pseudo-copies in $N_{\mathcal{P}}(v|t)$ in which some fixed vertex u appears. Clearly, we always have $0 \leq \text{conf}_{\mathcal{P}}(v|t) \leq 1$. If v and t are not compatible, then we set $\text{conf}_{\mathcal{P}}(v|t) = 0$. Globally, we define $\text{conf}(\mathcal{P}|t) := \max_v \text{conf}_{\mathcal{P}}(v|t)$ and $\text{conf}(\mathcal{P}) := \max_t \text{conf}(\mathcal{P}|t)$.

To ensure that the isomorphic copies in the T -pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$ are sufficiently spread out, we also require $\text{conf}(\mathcal{I}) \leq \delta$ for some given $\delta > 0$. In Section 3, we prove that such a T -pseudo-decomposition can always be obtained provided the minimum degree in each colour is large enough.

Lemma 10. *Let T be a tree on m edges and ε, δ real numbers with $0 < \varepsilon, \delta < 1$. Let G be a T -equitably coloured bipartite graph where the minimum degree in each colour is at least $(10m)^{18}(\varepsilon\delta)^{-6}$. Then G admits a T -pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$, where \mathcal{I} denotes the collection of isomorphic copies of T , such that:*

- (1) *for every compatible $v \in V(G)$ and $t \in V(T)$, we have $d_{\mathcal{H}}(v|t) \leq \varepsilon d_{\mathcal{I}}(v|t)$;*
- (2) $\text{conf}(\mathcal{I}) \leq \delta$.

Step 2: Repairing non-isomorphic copies

For this part of the proof we label the vertices t_0, \dots, t_m of T so that, for every $i \in \{1, \dots, m\}$, the subgraph induced by t_0, \dots, t_i is connected. Such an ordering can for example be obtained by applying a breadth-first search algorithm from some vertex t_0 of T . We also label the edges of T so that e_i denotes the edge joining t_i with $T[t_0, \dots, t_{i-1}]$ for every $i \in \{1, \dots, m\}$. To indicate at which place a pseudo-copy H fails to be isomorphic to T , we introduce the following definitions.

Definition 11. *Let H be a pseudo-copy of T , and let v_i denote the image of t_i in H for every $i \in \{0, \dots, m\}$. For $i \in \{1, \dots, m\}$, we say that H is **i -good** if the vertices v_0, \dots, v_i are pairwise distinct. If H is not i -good, then we say that H is **i -bad**.*

Note that since G does not have multiple edges, every pseudo-copy of T in G is 2-good. Moreover, since G is bipartite, every pseudo-copy of T in G is even 3-good.

The idea is to use isomorphic copies to repair the pseudo-trees that are not isomorphic to T . We start by considering all pseudo-trees in \mathcal{H} that are 4-bad. For each such H , we will find an isomorphic copy $f(H)$ in \mathcal{I} such that $H \cup f(H)$ can be written as the union of two 5-good pseudo-copies of T , say $H_1 \cup H_2$. We then remove H from \mathcal{H} and $f(H)$ from \mathcal{I} , and add $\{H_1, H_2\}$ to \mathcal{H} . The technical definition of this so-called *switch* is given below. We use this operation for all 4-bad pseudo-copies of T in \mathcal{H} . Let $\mathcal{H}' \cup \mathcal{I}'$ denote the resulting T -pseudo-decomposition, where \mathcal{I}' again contains only isomorphic copies of T and all pseudo-copies in \mathcal{H}' are 4-good. We repeat this step, this time repairing all 5-bad pseudo-copies in \mathcal{H}' by using isomorphic copies in \mathcal{I}' . We continue like this until we get a T -pseudo-decomposition in which all pseudo-copies are m -good and thus isomorphic to T .

To make sure that we can perform a switch between H and $f(H)$, we need $f(H)$ to satisfy certain properties. Let v_j denote the image of t_j in H for $j \in \{0, \dots, m\}$ and suppose i is chosen minimal such that H is i -bad. By the choice of our labelling, there exists $i' \in \{0, \dots, i-1\}$ with $t_{i'}t_i \in E(T)$. To ensure that v_i is distinct from the previous vertices v_0, \dots, v_{i-1} , we want to choose a different edge corresponding to e_i at $v_{i'}$. Since we take this edge from $f(H)$, we want $f(H)$ to also use the vertex $v_{i'}$ as image of t'_i . However, this should be the only point of intersection with H to ensure that both copies will be i -good after the switch.

More precisely, for every edge $e_i \in E(T)$, let T^{i-} denote the connected component of $T - e_i$ containing t_0 . Let T^{i+} be the subgraph of T induced by $E(T) \setminus E(T^{i-})$. If H is a pseudo-copy of T , then we denote the images of T^{i-} and T^{i+} under the homomorphism by H^{i-} and H^{i+} . Now we are ready to define the switching operation.

Definition 12. Let \mathcal{H} be a collection of pseudo-copies of T in G and $i \in \{1, \dots, m\}$. Let $t_{i'}$ be the endpoint of the edge e_i that is different from t_i . Suppose $H_1, H_2 \in N_{\mathcal{H}}(v|t_{i'})$ for some $v \in V(G)$. The i -**switch** of $\{H_1, H_2\}$ is defined by

$$\text{sw}_i(\{H_1, H_2\}) = \{H_1^{i+} \cup H_2^{i-}, H_1^{i-} \cup H_2^{i+}\}.$$

By making an i -switch between two pseudo-copies H and $f(H)$, their vertices corresponding to v_0, \dots, v_{i-1} remain unchanged. In particular, if both H and $f(H)$ are $(i-1)$ -good, then also both copies in $\text{sw}_i(\{H, f(H)\})$ will be $(i-1)$ -good. Moreover, if $H \cap f(H) = \{v_{i'}\}$, then after the switch both pseudo-trees will be i -good. Notice that neither of the two new pseudo-trees is necessarily still isomorphic to T . In particular, the collection of isomorphic copies might shrink with every step of the repairing process.

If the pseudo-trees in \mathcal{I} overlap too much, we might not be able to find a single pseudo-tree $f(H)$ in \mathcal{I} with $H \cap f(H) = \{v_{i'}\}$. A sufficiently low conflict ratio of \mathcal{I} ensures that we can find such a function $f : \mathcal{H} \rightarrow \mathcal{I}$. However, to continue this process we also need that the remaining collection of isomorphic copies $\mathcal{I} \setminus f(\mathcal{H})$ has a low conflict ratio. To this end we use the Local Lemma to prove the following lemma in Section 4.

Lemma 13. Let T be a tree on m edges and ε, δ positive real numbers with $\varepsilon + \delta m < \frac{1}{2}$ and $\varepsilon m < 1$. Let \mathcal{H} and \mathcal{H}' be collections of pseudo-copies of T in G with $\text{conf}(\mathcal{H}') \leq \delta$ and $d_{\mathcal{H}'}(v|t) > \max\{22/\varepsilon^7, d_{\mathcal{H}}(v|t)/\varepsilon\}$ for each compatible $v \in V(G)$, $t \in V(T)$. For every $t \in V(T)$, there exists an injective function $f_t : \mathcal{H} \rightarrow \mathcal{H}'$ such that

- $f_t(N_{\mathcal{H}}(v|t)) \subset N_{\mathcal{H}'}(v|t)$ for every $v \in V(G)$ compatible with t ,
- $H \cap f_t(H) = \{v\}$ for every $H \in N_{\mathcal{H}}(v|t)$, and
- $d_{f_t(\mathcal{H})}(v|t') \leq 3\varepsilon d_{\mathcal{H}'}(v|t')$ for every compatible $v \in V(G)$, $t' \in V(T)$.

By using Lemma 13 with $\mathcal{H}' = \mathcal{I}$, we find a collection $f(\mathcal{H})$ in which the degrees are low compared to the degrees in \mathcal{I} . Thus, the conflict ratio of the collection of isomorphic copies only increases by a constant factor after each step of the repairing process. By choosing ε and δ sufficiently small, the proof of Theorem 7 will follow from Lemma 10 and repeated applications of Lemma 13. The details can be found at the end of Section 4.

3 Finding a good T -pseudo-decomposition

Given a graph with a T -equitable colouring and large minimum degree in each colour, we construct a T -pseudo-decomposition satisfying the conditions in Lemma 10. As described in Step 1 of Section 2, every T -equitable colouring gives rise to several T -pseudo-decompositions. We form the pseudo-copies of T by grouping the edges at every vertex randomly into rainbow stars. If the degrees in each colour are large enough, we can ensure that most of the resulting pseudo-trees are isomorphic to T and also the conflict ratio of the resulting T -pseudo-decomposition is small. The proof of this is essentially an application of the Local Lemma.

Proposition 14 (Symmetric Local Lemma). *Let A_1, \dots, A_n be events in some probability space Ω with $\mathbb{P}[A_i] \leq p$ for all $i \in \{1, \dots, n\}$. Suppose that each A_i is mutually independent of all but at most d other events A_j . If $4pd < 1$, then $\mathbb{P}[\cap_{i=1}^n \overline{A_i}] > 0$.*

The bad events in this case are of the form that many copies in $N_{\mathcal{H}}(v|t)$ are either not isomorphic to T or have a vertex different from v in common. To show that each event occurs with low probability, we make use of an inequality due to McDiarmid [9] (see also [11]).

Proposition 15 (McDiarmid's Inequality (simplified version)). *Let X be a non-negative random variable, not identically 0, which is determined by m independent permutations Π_1, \dots, Π_m . A choice is the position that a particular element gets mapped to in a permutation. If there exist $d, r > 0$ such that*

- *interchanging two elements in any one permutation can affect X by at most d , and*
- *for any $s > 0$, if $X \geq s$ then there is a set of at most rs choices whose outcomes certify that $X \geq s$,*

then for any $0 \leq \lambda \leq \mathbb{E}[X]$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > \lambda + 60d\sqrt{r\mathbb{E}[X]}\right] \leq 4e^{-\frac{\lambda^2}{8d^2r\mathbb{E}[X]}}.$$

A necessary condition to apply the Local Lemma is that each event is mutually independent of most other events. To make sure that this is the case, we start the proof by partitioning the edges at each vertex into so-called *fans* of roughly the same size. Recall that for $v \in V(G)$ and $t \in V(T)$, we denote by $N_i(v)$ the edges coloured i incident with v in G , and by $S(t)$ the set of edges incident with t in T .

Proof of Lemma 10: Set $c = \lceil (10m)^9(\varepsilon\delta)^{-3} \rceil$. For every $v \in V(G)$ and colour i , we choose $r_{v,i} \in \{0, \dots, c-2\}$ such that $d_i(v) \equiv r_{v,i} \pmod{c-1}$. Since the minimum degree in each colour in G is greater than c^2 , we can partition every set $N_i(v)$ into subsets of size c and $c-1$ so that precisely $r_{v,i}$ of them have size c . We call these subsets *i-blades*. Note that an edge uv of colour i in G appears both in an i -blade of $N_i(u)$ as well as in an i -blade of $N_i(v)$, but we do not require these two i -blades to have the same size.

For every compatible $t \in V(T)$, $v \in V(G)$, and $i, j \in S(t)$, we have $d_i(v) = d_j(v)$ since the colouring is T -equitable. Thus, the number of i -blades of size c (respectively, of size $c-1$) in the partition of $N_i(v)$ is equal to the number of j -blades of size c (respectively, of size $c-1$) in the partition of $N_j(v)$. We can therefore partition the edges of $N_{S(t)}(v)$ into *fans*, which are unions of blades of the same size, such that every fan contains precisely one i -blade for every $i \in S(t)$. In other words, a fan φ at a vertex v (with relation to t) is a subset of $N_{S(t)}(v)$ of size $c|S(t)|$ or $(c-1)|S(t)|$ such that all colours in $S(t)$ appear c times or $c-1$ times in φ . We also call φ a *t-fan* to indicate the colours appearing in φ .

For every compatible $t \in V(T)$, $v \in V(G)$, and every t -fan φ at v , we uniformly at random partition the edges in φ into rainbow stars of size $|S(t)|$. More precisely, for every $i \in S(t)$ we choose a permutation $\Pi_{\varphi,i}$ independently and uniformly at random from all permutations on c elements (respectively, on $c-1$ elements if the blades of φ have size $c-1$). By labelling the edges of each blade, each permutation $\Pi_{\varphi,i}$ corresponds to an

ordering of the edges of the i -blade of φ . Now we partition the edges of φ into stars of size $|S(t)|$ by grouping the edges of different blades that were mapped to the same position. In other words, for every $s \in \{1, \dots, c\}$ (respectively, $s \in \{1, \dots, c-1\}$) we form a star by choosing for every $i \in S(t)$ the edge labelled $\Pi_{\varphi,i}(s)$ in the i -blade of φ . These stars are centered at v and each colour in $S(t)$ appears precisely once. Note that every edge $uv \in E(G)$ belongs to exactly two stars, one centered at u and one centered at v . As described in Step 1 in Section 2, these stars correspond to a T -pseudo-decomposition of G in a canonical way. All that remains to show is that there exists an outcome of the random permutations such that the resulting T -pseudo-decomposition satisfy the conditions (1) and (2) of Lemma 10.

We denote the set of pseudo-trees using edges of a fan φ by \mathcal{T}_φ . Note that $|\mathcal{T}_\varphi|$ is either equal to $c-1$ or c . Now we formally define what the bad events at a t -fan φ at a vertex v are. Let A_φ be the event that more than $2m^2c^{2/3}$ of the pseudo-copies in \mathcal{T}_φ are not isomorphic to T . Let B_φ be the event that there exists a vertex $u \in V(G)$ with $u \neq v$ such that more than $2mc^{2/3}$ pseudo-copies in \mathcal{T}_φ contain u . Finally, let $C_\varphi = A_\varphi \cup B_\varphi$. We will prove the following two statements.

Claim 1: Each C_φ is mutually independent of all but at most $4(cm)^{2m}$ other events C_ψ .

Claim 2: $\mathbb{P}[C_\varphi] < 9(cm)^m m e^{-c^{2/3}/32}$.

Before we proceed to prove these claims, let us note that they allow us to use the Local Lemma to get our desired T -pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$. Indeed, since $e^{-x} < \frac{(9m)!}{x^{9m}}$ for $x > 0$, we have

$$\begin{aligned}
4 \cdot 4(cm)^{2m} \cdot \mathbb{P}[C_\varphi] &< 2^8 \cdot (cm)^{3m} \cdot m \cdot e^{-c^{2/3}/32} \\
&< 2^{45m+8} \cdot \left(\frac{m}{c}\right)^{3m} \cdot m \cdot (9m)! \\
&< \left(2^{18} \cdot \frac{m}{c} \cdot (9m)^3\right)^{3m} \\
&< \left(\frac{10^9 m^4}{c}\right)^{3m} \\
&< 1,
\end{aligned}$$

where the last inequality follows from $c \geq (10m)^9$. Thus, the symmetric version of the Local Lemma yields a T -pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$ for which none of the events C_φ holds. Now $\mathcal{H} \cup \mathcal{I}$ has the desired properties:

- Since A_φ does not hold for any φ , at most $2m^2c^{2/3}$ of the pseudo-copies in \mathcal{T}_φ are not isomorphic to T . Since $c \geq (10m)^9 \varepsilon^{-3}$, we have $2m^2c^{2/3} < \frac{\varepsilon}{1+\varepsilon}c$. Thus, less than $\frac{\varepsilon}{1+\varepsilon}c$ of the pseudo-copies in \mathcal{T}_φ are in \mathcal{H} , while at least $\frac{1}{1+\varepsilon}c$ of them are in \mathcal{I} . This holds for every t -fan at v , so we have $d_{\mathcal{H}}(v|t) < \varepsilon d_{\mathcal{I}}(v|t)$.
- Since B_φ does not hold for any φ , there are at most $2mc^{2/3}$ trees in \mathcal{T}_φ containing a given vertex u different from v . As argued above, at least $\frac{c}{1+\varepsilon}$ of the pseudo-copies in \mathcal{T}_φ are in \mathcal{I} . Since $c \geq (10m)^9(\varepsilon\delta)^{-3}$, we have $2mc^{2/3} < \delta \frac{c}{1+\varepsilon}$. Thus, the proportion

of trees in $\mathcal{T}_\varphi \cap \mathcal{I}$ containing u is less than δ . This is true for every t -fan at v , so we have

$$\frac{|\{H \in N_{\mathcal{I}}(v|t) : u \in V(H)\}|}{d_{\mathcal{I}}(v|t)} \leq \delta$$

and thus $\text{conf}(\mathcal{I}) \leq \delta$.

It remains to verify Claims 1 and 2. We begin by proving Claim 1.

Proof of Claim 1: The structure of \mathcal{T}_φ depends on permutations in different fans. Let $J(\varphi)$ denote the set of fans ψ for which there exists an outcome of the random permutations such that $\mathcal{T}_\varphi \cap \mathcal{T}_\psi$ is non-empty. Since each fan consists of at most cm edges, there are at most $cm + (cm)^2 + \dots + (cm)^m$ fans we can reach from φ via a path of length at most m . Thus,

$$|J(\varphi)| \leq cm + (cm)^2 + \dots + (cm)^m < 2(cm)^m.$$

This shows that there are at most $2(cm)^m$ fans where the outcome of the permutation affects the structure of \mathcal{T}_φ . The same calculation shows that each permutation affects the structure of at most $2(cm)^m$ sets \mathcal{T}_ψ . Hence, the event C_φ is mutually independent of all but at most $4(cm)^{2m}$ other events C_ψ . \square

Before we prove Claim 2, let us introduce more terminology. Let t_i and t_j be two distinct vertices of T . Notice that t_i or t_j could be equal to t . We say that a pseudo-copy H of T is (t_i, t_j) -bad if the images of t_i and t_j in H are identical. For a t -fan φ at a vertex v , let $A_\varphi(t_i, t_j)$ be the event that the number of (t_i, t_j) -bad pseudo-trees in \mathcal{T}_φ is greater than $2c^{2/3}$. For a vertex $u \in V(G)$ with $u \neq v$, let $B_\varphi(u|t_i)$ be the event that the number of pseudo-trees in \mathcal{T}_φ in which u is the image of t_i is greater than $2c^{2/3}$. The proof of Claim 2 consists of two parts:

Claim 2A: $\mathbb{P}[A_\varphi(t_i, t_j)] < 4e^{-c^{2/3}/32}$ for every $t_i, t_j \in V(T)$ with $t_i \neq t_j$.

Claim 2B: $\mathbb{P}[B_\varphi(u|t_i)] < 4e^{-c^{2/3}/8}$ for every $u \in V(G)$, $t_i \in V(T)$ and $u \neq v$.

The proofs of Claims 2A and 2B use McDiarmid's inequality and have a very similar structure. We will therefore present all the details in the proof of Claim 2A, and only point out the differences in the proof of Claim 2B.

Proof of Claim 2A: Fix t_i and t_j as different vertices of T . Let P_i and P_j denote the paths in T from t to t_i and t_j . In the case that one is a subpath of the other, we may assume that P_i is contained in P_j . Let $t_{j'}$ denote the second last vertex of P_j and let j denote the edge joining $t_{j'}$ and t_j . Now $T - j$ consists of two components, one of which contains t_j while the other one contains t , t_i , and $t_{j'}$.

Let π be a fixed outcome of all permutations apart from those at the j -blades of $t_{j'}$ -fans. In other words, given π , we only need to know the outcome of the permutations $\Pi_{\psi,j}$ for every $t_{j'}$ -fan ψ to construct the T -pseudo-decomposition. For any such outcome π , we will show that the conditional probability $\mathbb{P}[A_\varphi(t_i, t_j)|\pi]$ is at most $4e^{-c^{2/3}/32}$. Clearly, since we condition on an arbitrary but fixed event, this uniform bound implies Claim 2A.

Let T' denote the component of $T - j$ containing t , t_i and $t_{j'}$, and let T'' denote the subgraph of T induced by $E(T) \setminus E(T')$. Let \mathcal{T}'_φ denote the images of T' in the pseudo-trees of \mathcal{T}_φ . By fixing π , the set \mathcal{T}'_φ is also fixed. The permutations of the j -blades at the $t_{j'}$ -fans only decide how the images of T' and T'' get matched at the $t_{j'}$ -fans.

Let Ψ denote the set of t_j -fans which contain edges of pseudo-copies in \mathcal{T}_φ . Note that also the set Ψ is completely determined by π . Let X_φ denote the random variable counting the number of (t_i, t_j) -bad pseudo-trees in \mathcal{T}_φ conditional on π . Notice that X_φ only depends on the random permutations $\Pi_{\psi,j}$ with $\psi \in \Psi$.

For each pseudo-tree $H \in \mathcal{T}'_\varphi$ at a t_j -fan $\psi \in \Psi$, we already know what the image of t_i in H is. There are $c-1$ or c different images of T'' that could get matched to H at ψ , each having a distinct vertex as image of t_j . Thus, there are at least $c-1$ different vertices that could be the image of t_j in H . Since the permutation $\Pi_{\psi,j}$ is chosen uniformly at random, the probability that H will be part of a (t_i, t_j) -bad pseudo-tree is at most $\frac{1}{c-1}$. Now, by linearity of expectation,

$$\mathbb{E}[X_\varphi] \leq |\mathcal{T}_\varphi| \cdot \frac{1}{c-1} \leq \frac{c}{c-1}.$$

We will apply McDiarmid's inequality to the random variable Y_φ defined by $Y_\varphi := X_\varphi + c^{2/3}$. Clearly $\mathbb{E}[Y_\varphi] = \mathbb{E}[X_\varphi] + c^{2/3}$. Only the permutations $\Pi_{\psi,j}$ with $\psi \in \Psi$ affect X_φ and thus Y_φ . If two elements in one of these permutations are interchanged, then the structure of two pseudo-trees in \mathcal{T}_φ changes. In particular, the number of (t_i, t_j) -bad trees in \mathcal{T}_φ changes by at most 2. Thus, we can choose $d = 2$ in McDiarmid's inequality.

If $Y_\varphi \geq s$, then $X_\varphi \geq s - c^{2/3}$, and thus at least $s - c^{2/3}$ of the pseudo-trees in \mathcal{T}_φ are (t_i, t_j) -bad. Let $H' \in \mathcal{T}'_\varphi$ be a part of a pseudo-tree H that is counted by X_φ . Let v_i and v_j denote the images of t_i and t_j in H . To verify that H is (t_i, t_j) -bad, we only need to know which edge in the j -blade of ψ gets mapped to the same position as the edges in H' in other blades of ψ . In other words, the vertex v_j is determined by the position of one element in the permutation $\Pi_{\psi,j}$, and thus $v_i = v_j$ can be certified by a single outcome. Thus, $X_\varphi \geq s - c^{2/3}$ can be certified by the outcomes of $s - c^{2/3} < s$ choices and we can choose $r = 1$ in McDiarmid's inequality.

By applying McDiarmid's inequality to Y_φ with $\lambda = \mathbb{E}[Y_\varphi]$, $d = 2$, $r = 1$, we get

$$\mathbb{P} \left[|Y_\varphi - \mathbb{E}[Y_\varphi]| > \mathbb{E}[Y_\varphi] + 120\sqrt{\mathbb{E}[Y_\varphi]} \right] \leq 4e^{-\frac{\mathbb{E}[Y_\varphi]}{32}} \leq 4e^{-\frac{c^{2/3}}{32}}.$$

Since $c \geq 10^9$ and $\mathbb{E}[Y_\varphi] \leq c^{2/3} + \frac{c}{c-1}$, we have $120\sqrt{\mathbb{E}[Y_\varphi]} < \frac{1}{2}\mathbb{E}[Y_\varphi]$ which implies

$$\mathbb{P} \left[X_\varphi > 2c^{2/3} \right] = \mathbb{P} \left[Y_\varphi > 3c^{2/3} \right] < \mathbb{P} \left[|Y_\varphi - \mathbb{E}[Y_\varphi]| > \frac{3}{2}\mathbb{E}[Y_\varphi] \right] \leq 4e^{-\frac{c^{2/3}}{32}}.$$

Now $\mathbb{P}[A_\varphi(t_i, t_j) | \pi] < 4e^{-c^{2/3}/32}$ and Claim 2A follows. \square

Proof of Claim 2B: Let $t_i \in V(T)$ be a fixed vertex different from t . Let P denote the path from t to t_i in T . Let t_j denote the second last vertex of P and let i denote the edge joining t_j and t_i . Now $T - i$ consists of two components, one of which contains t and t_j while the other one contains t_i . Let π be any fixed outcome of all permutations apart from those at the i -blades of t_j -fans. We show that the conditional probability $\mathbb{P}[B_\varphi(u|t_i) | \pi]$ is at most $4e^{-c^{2/3}/8}$. As π is arbitrary, this implies the general bound $\mathbb{P}[B_\varphi(u|t_i)] < 4e^{-c^{2/3}/8}$.

Let X_φ denote the random variable conditional on π which counts the number of pseudo-trees in \mathcal{T}_φ where u is the image of t_i . The vertex u appears at most once in each t_j -fan, so by linearity of expectation we have

$$\mathbb{E}[X_\varphi] \leq |\mathcal{T}_\varphi| \cdot \frac{1}{c-1} \leq \frac{c}{c-1}.$$

We apply McDiarmid's inequality to the random variable $X_\varphi + c^{2/3}$. Swapping two positions in a permutation $\Pi_{\psi,i}$ can affect X_φ by at most 1 since u is incident to at most one edge of the i -blade of ψ . If $X_\varphi + c^{2/3} \geq s$, then this can be certified by revealing at most s positions in the random permutations. Thus, applying McDiarmid's inequality to the random variable $X_\varphi + c^{2/3}$ with $\lambda = \mathbb{E}[X_\varphi] + c^{2/3}$, $r = 1$, $d = 1$ yields

$$\mathbb{P}[X_\varphi > 2c^{2/3}] \leq 4e^{-c^{2/3}/8}.$$

Now $\mathbb{P}[B_\varphi(u|t_i)|\pi] < 4e^{-c^{2/3}/8}$ and Claim 2B follows. \square

Now the proof of Claim 2 follows easily from Claims 2A and 2B.

Proof of Claim 2: By Claim 2A, we have

$$\mathbb{P}[A_\varphi] \leq \mathbb{P}\left[\bigcup_{\forall i < j} A_\varphi(t_i, t_j)\right] \leq \sum_{\forall i < j} \mathbb{P}[A_\varphi(t_i, t_j)] < 4m^2 e^{-c^{2/3}/32}.$$

Let $B_\varphi(u)$ be the event that the number of pseudo-trees in \mathcal{T}_φ containing u is greater than $2mc^{2/3}$. Since u cannot be the image of t , we have, by Claim 2B,

$$\mathbb{P}[B_\varphi(u)] \leq \mathbb{P}\left[\bigcup_{\forall i} B_\varphi(u|t_i)\right] \leq \sum_{\forall i} \mathbb{P}[B_\varphi(u|t_i)] < 4me^{-c^{2/3}/8}.$$

Since each fan consists of at most cm edges, there are at most $cm + (cm)^2 + \dots + (cm)^m$ vertices we can reach from φ via a path of length at most m . Thus, there are less than $2(cm)^m$ vertices u for which $\mathbb{P}[B_\varphi(u)]$ could be positive. In particular, we have

$$\mathbb{P}[B_\varphi] = \mathbb{P}\left[\bigcup_{\forall u, u \neq v} B_\varphi(u)\right] \leq \sum_{\forall u, u \neq v} \mathbb{P}[B_\varphi(u)] < 8(cm)^m me^{-c^{2/3}/8}$$

and Claim 2 follows from $\mathbb{P}[C_\varphi] \leq \mathbb{P}[A_\varphi] + \mathbb{P}[B_\varphi]$. \square

This concludes the proof of Lemma 10. \square

4 Repairing non-isomorphic copies

Let $\mathcal{H} \cup \mathcal{I}$ be the T -pseudo-decomposition given by Lemma 10. As described in Step 2 in Section 2, we use copies in \mathcal{I} to repair the pseudo-trees in \mathcal{H} that are not isomorphic to T . We apply Lemma 13 to show the existence of a suitable subset of \mathcal{I} to perform the switches. The proof of Lemma 13 relies on the following probabilistic tools, see also [11].

Proposition 16 (Lovász Local Lemma). *Let A_1, \dots, A_n be a finite set of events in some probability space Ω , and suppose that for some $J_i \subset [n]$, A_i is mutually independent of $\{A_j : j \notin J_i \cup \{i\}\}$. If there exist real numbers x_1, \dots, x_n in $(0, 1)$ such that $\mathbb{P}[A_i] \leq x_i \prod_{j \in J_i} (1 - x_j)$ for every $i \in \{1, \dots, n\}$, then $\mathbb{P}[\bigcap_{i=1}^n \overline{A_i}] > 0$.*

Proposition 17 (Simple Concentration Bound). *Let X be a random variable determined by n independent trials T_1, \dots, T_n such that changing the outcome of any one trial T_i can affect X by at most c . Then*

$$\mathbb{P}[|X - \mathbb{E}[X]| > \lambda] \leq 2e^{-\lambda^2/(2c^2n)}.$$

Proof of Lemma 13: Consider a pseudo-tree $H \in N_{\mathcal{H}}(v|t)$, and let $u \in V(H) \setminus \{v\}$. Since $\text{conf}(\mathcal{H}'|t) \leq \delta$, there are no more than $\delta d_{\mathcal{H}'}(v|t)$ trees in $N_{\mathcal{H}'}(v|t)$ containing u . Thus, there are at least $(1 - \delta m) d_{\mathcal{H}'}(v|t)$ pseudo-copies of T in $N_{\mathcal{H}'}(v|t)$ that intersect H only in v . Since $d_{\mathcal{H}}(v|t) \leq \varepsilon d_{\mathcal{H}'}(v|t)$, we can associate a set $S(H)$ of $\lfloor \frac{1-\delta m}{\varepsilon} \rfloor$ pseudo-copies in $N_{\mathcal{H}'}(v|t)$ with each $H \in N_{\mathcal{H}}(v|t)$ such that each element of $N_{\mathcal{H}'}(v|t)$ is contained in at most one of these sets. We define the function f_t by choosing $f_t(H)$ uniformly at random from one of the pseudo-trees in $S(H)$. Clearly, any such function will satisfy the first two conditions of Lemma 13. All that remains to show is that with positive probability $d_{f_t(\mathcal{H})}(v|t') \leq 3\varepsilon d_{\mathcal{H}'}(v|t')$ holds for every compatible $v \in V(G)$, $t' \in V(T)$.

The value of $d_{f_t(\mathcal{H})}(v|t')$ only depends on the set of pseudo-trees in $N_{\mathcal{H}'}(v|t')$ that are contained in some $S(H)$. Let \mathcal{H}'' be the collection of pseudo-copies of \mathcal{H}' that are contained in some $S(H)$. Clearly, each tree in $N_{\mathcal{H}''}(v|t')$ can be matched with exactly one tree in \mathcal{H} and this occurs with probability $\lfloor \frac{1-\delta m}{\varepsilon} \rfloor^{-1}$. By linearity of expectation,

$$\mathbb{E}[d_{f_t(\mathcal{H})}(v|t')] = \left\lfloor \frac{1-\delta m}{\varepsilon} \right\rfloor^{-1} d_{\mathcal{H}''}(v|t') < 2\varepsilon d_{\mathcal{H}'}(v|t').$$

Let $A_{v,t'}$ be the event that $d_{f_t(\mathcal{H})}(v|t') > 3\varepsilon d_{\mathcal{H}'}(v|t')$. Note that $d_{f_t(\mathcal{H})}(v|t')$ is completely determined by at most $d_{\mathcal{H}''}(v|t')$ independent trials. Since the outcome of each trial can affect $d_{f_t(\mathcal{H})}(v|t')$ by at most 1, the Simple Concentration Bound gives

$$\mathbb{P}[A_{v,t'}] < 2e^{-\varepsilon^2 d_{\mathcal{H}''}(v|t')/2}.$$

We claim that $A_{v,t'}$ is mutually independent of all but at most $m \lfloor \frac{1-\delta m}{\varepsilon} \rfloor d_{\mathcal{H}''}(v|t')$ other events $A_{v',t''}$. Indeed, $A_{v,t'}$ depends on at most $d_{\mathcal{H}''}(v|t')$ random trials, and in each trial we have a choice of $\lfloor \frac{1-\delta m}{\varepsilon} \rfloor$ trees to match. Each tree affects precisely m events other than $A_{v,t'}$.

Now we apply Proposition 16 to show that with positive probability none of the events $A_{v,t'}$ occur. Set $x = \frac{\varepsilon^4}{8}$. It is sufficient to show that

$$x(1-x)^{m \lfloor \frac{1-\delta m}{\varepsilon} \rfloor d_{\mathcal{H}''}(v|t')} \geq \mathbb{P}[A_{v,t'}]$$

holds for all compatible $v \in V(G)$, $t' \in V(T)$. If $d_{\mathcal{H}''}(v|t') < (\frac{2}{\varepsilon})^6$, then $d_{f_t(\mathcal{H})}(v|t') < (\frac{2}{\varepsilon})^6 < 3\varepsilon d_{\mathcal{H}'}(v|t')$, so $\mathbb{P}[A_{v,t'}] = 0$. If $d_{\mathcal{H}''}(v|t') \geq (\frac{2}{\varepsilon})^6$, then we have

$$\begin{aligned} x(1-x)^{m \lfloor \frac{1-\delta m}{\varepsilon} \rfloor d_{\mathcal{H}''}(v|t')} &\geq x(1-x)^{d_{\mathcal{H}''}(v|t')/\varepsilon^2} \\ &\geq x e^{-2x d_{\mathcal{H}''}(v|t')/\varepsilon^2} \\ &\geq \mathbb{P}[A_{v,t'}] \cdot \frac{x}{2} \cdot e^{\frac{\varepsilon^2}{4} d_{\mathcal{H}''}(v|t')} \\ &\geq \mathbb{P}[A_{v,t'}] \cdot \left(\frac{\varepsilon}{2}\right)^6 d_{\mathcal{H}''}(v|t') \\ &\geq \mathbb{P}[A_{v,t'}]. \end{aligned}$$

By the Local Lemma, there is a positive probability that none of the bad events occur. Thus, there exists a function f_t with the desired properties. \square

We now have all ingredients for the proof of Theorem 7.

Proof of Theorem 7: As described in Step 2 of Section 2, let t_0, \dots, t_m be a labelling of the vertices of T such that $T[t_0, \dots, t_i]$ is connected for every $i \in \{1, \dots, m\}$. We also label the edges of T so that e_i denotes the edge joining t_i with $T[t_0, \dots, t_{i-1}]$ for every $i \in \{1, \dots, m\}$. Set $\varepsilon_i = 5^{i-m}/15m$ for $i \in \{1, \dots, m\}$. We are going to construct a sequence $(\mathcal{H}_i \cup \mathcal{I}_i)_{i=1}^m$ of T -pseudo-decompositions of G such that the following holds:

- \mathcal{I}_i is a collection of isomorphic copies of T for every $i \in \{1, \dots, m\}$;
- \mathcal{H}_i is i -good for every $i \in \{1, \dots, m\}$;
- $d_{\mathcal{I}_i}(v|t) > \max\{22/\varepsilon_i^7, d_{\mathcal{H}_i}(v|t)/\varepsilon_i\}$ for every compatible $v \in V(G)$, $t \in V(T)$;
- $\text{conf}(\mathcal{I}_i) \leq \varepsilon_i$ for every $i \in \{1, \dots, m\}$.

Since the minimum degree in each colour in G is at least 10^{50m} , we can apply Lemma 10 with parameters $\varepsilon = \delta = 10^{-2m}$. Let $\mathcal{H} \cup \mathcal{I}$ denote the resulting T -pseudo-decomposition. Clearly $\mathcal{H} \cup \mathcal{I}$ satisfies the conditions for $\mathcal{H}_1 \cup \mathcal{I}_1$. Let $i \in \{2, \dots, m\}$ and suppose we have constructed $\mathcal{H}_{i-1} \cup \mathcal{I}_{i-1}$ such that the conditions above are satisfied. We need to repair the pseudo-trees in \mathcal{H}_{i-1} that are not i -good. Since the pseudo-trees in \mathcal{H}_{i-1} are all $(i-1)$ -good, we can achieve this by making i -switches. Let t_j be the endpoint of e_i that is different from t_i . Let $f_j : \mathcal{H}_{i-1} \rightarrow \mathcal{I}_{i-1}$ be the function we get by applying Lemma 13 with $\mathcal{H} = \mathcal{H}_{i-1}$, $\mathcal{H}' = \mathcal{I}_{i-1}$, $\varepsilon = \delta = \varepsilon_{i-1}$, and $t = t_j$. Now $f_j(\mathcal{H}_{i-1})$ is the set of trees we use to repair the pseudo-trees in \mathcal{H}_{i-1} that are not i -good. Set

$$\begin{aligned} \mathcal{H}_i &= \bigcup_{H \in \mathcal{H}_{i-1}} \text{sw}_i(H, f_j(H)) \text{ and} \\ \mathcal{I}_i &= \mathcal{I}_{i-1} \setminus f_j(\mathcal{H}_{i-1}), \end{aligned}$$

where $\text{sw}_i(H, f_j(H))$ denotes the i -switch of H and $f_j(H)$ as defined in Section 2. Since $H \cap f_j(H) = \{v\}$ for every $H \in N_{\mathcal{H}_{i-1}}(v|t_j)$, the two pseudo-copies in $\text{sw}_i(H, f_j(H))$ are both i -good.

Notice that the degree $d_{\mathcal{H}_i}(v|t)$ of a vertex is invariant under i -switches between pseudo-trees in \mathcal{H}_i . Since $d_{f_j(\mathcal{H}_{i-1})}(v|t) \leq 3\varepsilon_{i-1}d_{\mathcal{I}_{i-1}}(v|t)$ holds for compatible $v \in V(G)$ and $t \in V(T)$, we have $d_{\mathcal{I}_i}(v|t) \geq (1 - 3\varepsilon_{i-1})d_{\mathcal{I}_{i-1}}(v|t)$ and $d_{\mathcal{H}_i}(v|t) \leq 4\varepsilon_{i-1}d_{\mathcal{I}_{i-1}}(v|t)$. Thus,

$$d_{\mathcal{H}_i}(v|t) \leq \frac{4\varepsilon_{i-1}}{(1 - 3\varepsilon_{i-1})}d_{\mathcal{I}_i}(v|t) \leq 5\varepsilon_{i-1}d_{\mathcal{I}_i}(v|t) = \varepsilon_i d_{\mathcal{I}_i}(v|t),$$

$$d_{\mathcal{I}_i}(v|t) \geq (1 - 3\varepsilon_{i-1})d_{\mathcal{I}_{i-1}}(v|t) \geq 22 \frac{1 - 3\varepsilon_{i-1}}{\varepsilon_{i-1}^7} \geq \frac{22}{\varepsilon_i^7}$$

and

$$\text{conf}(\mathcal{I}_i) \leq \frac{\text{conf}(\mathcal{I}_{i-1})}{1 - 3\varepsilon_{i-1}} \leq \frac{5}{4}\varepsilon_{i-1} < \varepsilon_i.$$

Hence, the T -pseudo-decomposition $\mathcal{H}_i \cup \mathcal{I}_i$ has the desired properties. In particular, \mathcal{H}_m is m -good and $\mathcal{H}_m \cup \mathcal{I}_m$ is a T -decomposition of G . \square

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